Prasad: six sided dice. $P = \{1, 2, 3, 4, 5, 6\}$.

Jim: — " — $J = \{1, 2, 3, 4, 5, 6\}$.

Ex. 1) $\Omega = \overset{P}{\{1, 2, \ldots 6\}} \times \overset{J}{\{1, 2, \ldots 6\}}$.

$= \{ \underline{11}, \underline{12}, 13, 14, \underline{15}, 16$

$\vdots \quad \underline{24}$ ←

$\vdots$

$61, 62, 63, 64, 65, 66\}$.

~~"Ω" = { Prasad gets 1, Prasad gets 2, . . . Prasad gets 6, Jim gets 1, Jim gets 2, . . . Jim gets 6}~~

Prasad gets 1 $= \{11, 12, 13, 14, 15, 16\}$.

Either. Prasad gets 1 or Jim gets 1 =

$\{11, 12, 13, 14, 15, 16\} \cup \{11, 21, 31, \ldots 61\}$.

$= \{11, 12, 13, 14, 15, 16, 21, 31, 41, 51, 61\}$.

(Prasad + Jim = 6.) $= \{15, 24, 33, 42, 51\}$.

Events are subsets of Ω.

In this example Ω has size 36.

Notation. $|\Omega| = 36$.

Number of possible subsets of $\Omega = 2^{36}$

Set of all possible subsets of $\Omega = 2^{\Omega}$ = Power set of Ω.

$= \{ \{\}, \{11\}, \{12\}, \ldots$

$\{11, 12\} \ldots \}$.

Ex 2. Coin. $\Omega = \{\text{Heads}, \text{Tails}\}$.

Power set of $\Omega = 2^{\Omega}$ = Set of all possible subsets of Ω.

$= \{ \{\} \quad \{\text{Heads}\} \quad \{\text{Tails}\}. \quad \{\text{Heads}, \text{Tails}\} \}$.

Ex 3. Prasad tosses a coin. $\Omega = \{\text{Heads}, \text{Tails}\} \times \{\text{Heads}, \text{Tail}\}$.
Jim tosses a coin. $= \{HH, HT, TH, TT\}$.

Power set of $\Omega = 2^{\Omega}$

$$= \{ \{\}, \underbrace{\{HH\}, \{HT\}, \{TH\}, \{TT\}}_{\binom{4}{1}},$$
$$\underbrace{\{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}}_{\binom{4}{2}},$$
$$\underbrace{\{HT, TH, TT\}, \{HH, TH, TT\}, \{HH, HT, TT\}, \{HH, HT, TH\}}_{\binom{4}{3}},$$
$$\underbrace{\{HH, HT, TH, TT\}}_{\binom{4}{4}} \}.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 = 16.$$ Binomial Theorem.

$$(x+y)^4 = \binom{4}{0} x^4 y^0 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} x^0 y^4$$

(2) Event space (σ-algebra) $\mathcal{E}$

* If Ω is finite / countable. $\mathcal{E} = 2^{\Omega}$

$\rightarrow \mathbb{N} = \{0, 1, 2, \ldots\}$. Discrete Probability Space.

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.

* If Ω is uncountable. $\mathcal{E}$ cannot be 2^{Ω} *

$\mathbb{R}$ = set of real numbers. Continuous Probability Space.

Closed $[0, 1]$: interval between 0 and 1 (incl. the end pts).

$(0, 1]$: $0 < x \leq 1$.

$[0, 1)$: $0 \leq x < 1$

open $(0, 1)$: $0 < x < 1$

Subsets of Ω.

$\mathbb{P}$: Probability assignment.

Collection of all events

$$\mathbb{P} : \mathcal{E} \longrightarrow [0, 1].$$

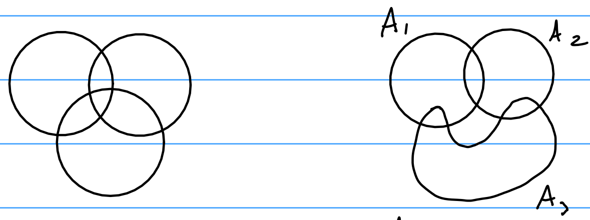
$\mathbb{P}$ is a function, takes every element of $\mathcal{E}$ and assigns it a real number in $[0, 1]$.

$\mathbb{P} : \mathcal{E} \to [0,1]$ must satisfy the following axioms:

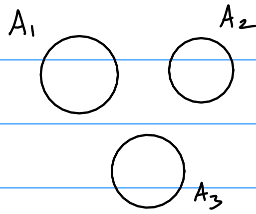
1. $\mathbb{P}(\Omega) = 1$.
2. If $A_1, A_2 \subset \Omega$ and $\underbrace{A_1 \cap A_2 = \{\}}_{\text{disjoint}}$ $(A_1, A_2 \in \mathcal{E})$.

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2).$$

If $A_1, A_2, A_3 \subset \Omega$ $(A_1, A_2, A_3 \in \mathcal{E})$



$A_1 \cap A_2 \cap A_3 = \{\}$.



$A_1 \cap A_2 = \{\}$ $A_1 \cap A_3 = \{\}$.

$A_2 \cap A_3 = \{\}$ <u>disjoint</u>

If $A_1, A_2, A_3, \ldots A_n \subset \Omega$ $(\in \mathcal{E})$, <u>$A_1 \ldots A_n$ are disjoint.</u>

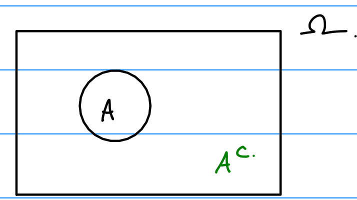
$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

$\downarrow$

* If $A_1, A_2, A_3, \ldots \subset \Omega$ $(\in \mathcal{E})$, (A_i disjoint)

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

* For all events A, $0 \le \mathbb{P}(A) \le 1$



* $A \cup A^c = \Omega$

A, A^c are disjoint.

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c)$$
$$= \mathbb{P}(A) + \mathbb{P}(A^c)$$

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

Discrete Probability Assignment

$\Omega = \{ HH, HT, TH, TT \}$.

Prasad's coin   Jim's coin.

Power set of $\Omega = 2^{\Omega}$

$\mathcal{E} = 2^{\Omega} = \{ \{\}, \{HH\}, \{HT\}, \{TH\}, \{TT\},$
$\{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\} \{HT, TT\}, \{TH, TT\},$
$\{HT, TH, TT\} \quad \{HH, TH, TT\}, \{HH, HT, TT\}, \{HH, HT, TH\},$
$\{HH, HT, TH, TT\} \}$.

$\rightarrow P\{HH\} = 1/3$

$\rightarrow P\{HT\} = 1/6$

$P\{TH\} = 1/6$

$P\{TT\} = 1/3$

$\mathbb{P}\{\underbrace{HH, HT}_{\{HH\} \cup \{HT\}}\} = \mathbb{P}\{HH\} + \mathbb{P}\{HT\}$.

$= 1/3 + 1/6 = 1/2$

$\mathbb{P}(\text{Prasad throws heads}) = \mathbb{P}\{HH, HT\} = 1/2$.

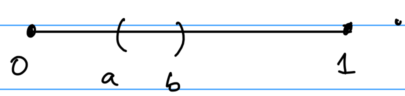
$\mathbb{P}\{HH, TH, TT\} = \mathbb{P}\{HH\} + \mathbb{P}\{TH\} + \mathbb{P}\{TT\}$.

$= 1/3 + 1/6 + 1/3 = 5/6$.

$\mathbb{P}\{\} = 0$.

Continuous Probability Assignments.

$\Omega = [0, 1]$



(a, b), $[a, b)$, $(a, b]$, $[a, b]$

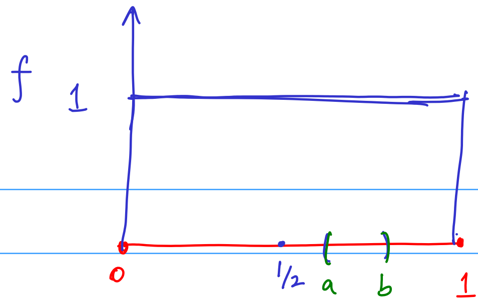
$[a, a] = \{a\}$.

$[0, 1/2] =$ set of all numbers between 0 and $1/2$.

$= \bigcup_{0 \le x \le 1/2} \{x\}$. (Axioms do not help)

$f : \Omega \to \mathbb{R}^+$

pdf



$$\int_\Omega f = 1.$$

$$\mathbb{P}[a,b] = \mathbb{P}(a,b) = \mathbb{P}[a,b) = \mathbb{P}(a,b] = \int_a^b f(x)\,dx.$$

$$\mathbb{P}[a,a] = \mathbb{P}\{a\} = \int_a^a f(x)\,dx = 0$$