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ECE 342: Probability and Statistics **Spring 2025**

Module 9.2: Markov Chains – Steady-State Behavior

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Read BT Chapter 7.3.

Learning Objectives:

- Know how to compute the steady-state distribution of a Markov chain.

9.1 Steady-State Behavior

We define a row vector $\boldsymbol{\pi}(n)$ as the PMF of X_n at time n:

$$\begin{aligned}\boldsymbol{\pi}(n) &= \begin{bmatrix}\pi_1(n) & \pi_2(n) & \cdots & \pi_m(n)\end{bmatrix} \\ &= \begin{bmatrix}P(X_n = 1) & P(X_n = 2) & \cdots & P(X_n = m)\end{bmatrix}\end{aligned}$$

Using the total probability theorem, we have the following state transition dynamics:

$$\boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n) \cdot \mathbf{R},$$

where $\mathbf{R}$ is the $m \times m$ state transition matrix.

Hence, from the initial state distribution $\boldsymbol{\pi}(0)$ at time 0, the state distribution at time n can be calculated as

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \cdot \mathbf{R}^n.$$

Recall the example of a student keeping up-to-date and falling behind. The transition probability matrix is

$$\mathbf{R} = \begin{bmatrix}0.8 & 0.2 \\ 0.6 & 0.4\end{bmatrix}.$$

We have

$$\mathbf{R}^2 = \begin{bmatrix}0.76 & 0.24 \\ 0.72 & 0.28\end{bmatrix}, \quad \mathbf{R}^3 = \begin{bmatrix}0.752 & 0.248 \\ 0.744 & 0.256\end{bmatrix}, \quad \mathbf{R}^4 = \begin{bmatrix}0.7504 & 0.2496 \\ 0.7488 & 0.2512\end{bmatrix}, \quad \mathbf{R}^5 = \begin{bmatrix}0.7501 & 0.2499 \\ 0.7498 & 0.2502\end{bmatrix}.$$

You can probably see the trend: the n-step transition matrix $\mathbf{R}^n$ is converging to

$$\lim_{n\to\infty} \mathbf{R}^n = \begin{bmatrix}0.75 & 0.25 \\ 0.75 & 0.25\end{bmatrix}.$$

In particular, the matrix at the limit has two identical rows. In fact, the row is the steady-state distribution, or stationary distribution.

Steady-State / Stationary Distribution

Under certain conditions (omitted here), a Markov chain can have a steady-state / stationary distribution $\boldsymbol{\pi} = [\pi_1, \ldots, \pi_m]$, which is the unique solution to the following equations:

$$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot \mathbf{R}$$
$$\sum_{i=1}^{m} \pi = 1$$

For the example above, we can solve for the stationary distribution $\boldsymbol{\pi}$ as follows:

$$\begin{bmatrix}\pi_1 & \pi_2\end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2\end{bmatrix} \cdot \begin{bmatrix}0.8 & 0.2 \\ 0.6 & 0.4\end{bmatrix},$$
$$\pi_1 + \pi_2 = 1,$$

which leads to three linear equations

$$\begin{aligned} \pi_1 &= 0.8\pi_1 + 0.6\pi_2 \\ \pi_2 &= 0.2\pi_1 + 0.4\pi_2 \\ 1 &= \pi_1 + \pi_2. \end{aligned}$$

Note that there are three equations and two variables. But the first two equations are linearly dependent. So we would only need two equations to solve for $\boldsymbol{\pi}$:

$$\begin{aligned} \pi_1 &= 0.8\pi_1 + 0.6\pi_2 \\ 1 &= \pi_1 + \pi_2, \end{aligned}$$

which gives us

$$\boldsymbol{\pi} = \begin{bmatrix}\pi_1 & \pi_2\end{bmatrix} = \begin{bmatrix}0.75 & 0.25\end{bmatrix}.$$

This is exactly the row in the matrix that $\mathbf{R}^n$ converges to!