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ECE 342: Probability and Statistics **Spring 2025**

Module 9.1: Markov Chains – Definitions and Examples

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Read BT Chapter 7.1 and 7.2.

Learning Objectives:

- Understand the Markov property;
- Know how to define a Markov chain, draw the transition probability graph, and write the transition probability matrix.
- Understand recurrent and transient states in a Markov chain.

9.1 Definition of Markov Chains

A (discrete-time) random process or stochastic process is a series of random variables at (discrete) time steps

$$\{X_n,\, n = 0, 1, 2, \ldots\},$$

where X_t is the random variable at time t. For example, the hourly temperature is a random process.

In general, the temporal dependence of the random variables is described by the conditional probability

$$P(X_{n+1} = j \,|\, X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0).$$

A discrete-time Markov chain is a special random process that satisfies the **Markov property** and the **stationary property**:

$$\begin{aligned} & P(X_{n+1} = j \,|\, X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) \\ = \; & P(X_{n+1} = j \,|\, X_n = i) \qquad \textit{(Markov property)} \\ = \; & p_{ij} \qquad \textit{(stationary property)} \end{aligned}$$

Markov property: the future X_{n+1} depends on the present X_n, but not on the past $X_{n-1}, \ldots, X_0$.

Stationary property: the transition probabilities p_{ij} is independent of the time step n.

9.2 Transition Probability Graph and Matrix

A Markov chain is completely described by its transition probability graph or transition probability matrix.

Consider a scenario where a student is taking a class and may be up-to-date or fall behind at each week (Example 7.1 in the book). We can use X_n to denote her state at week n, with $X_n = 1$ meaning "up-to-date" in week n and $X_n = 2$ meaning "behind" in week n. Suppose that if she is up-to-date this week, she will be up-to-date (or fall behind) with probability 0.8 (or 0.2), and if she falls behind this week, she will be up-to-date (or fall behind) with probability 0.6 (or 0.4).

This two-state Markov chain can be represented by the transition probability graph shown in Fig. 9.1.

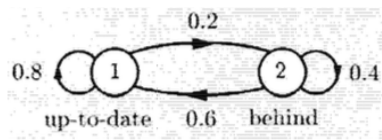

Figure 9.1: Illustration of a transition probability graph (Figure 7.1 in the book).

In the transition probability graph, there are two components: (1) nodes that represent the states, and (2) arrows that represent state transitions with transition probabilities annotated.

Alternatively, we can represent this two-state Markov chain by a 2×2 transition probability matrix

$$\mathbf{R} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix},$$

where the element on the ith row and jth column, p_{ij}, is the transition probability from state i to state j.

The transition probability graph and transition probability matrix are equivalent. The graph is more intuitive, while the matrix is more convenient to use in computation.

9.3 Classification of States

There can be up to two types of states in a Markov chain:

- Recurrent states: State i is recurrent, if from any state j that is accessible from i, state i is also accessible from j.
- Transient states: State i is transient if it is not recurrent.

Basically, if we are in a recurrent state, we may transit to other states, but will eventually revisit this state at some point in time. In contrast, if we are in a transient state, we may never come back to this state again.

To make it more clear, we can analyze the Markov chain shown below:

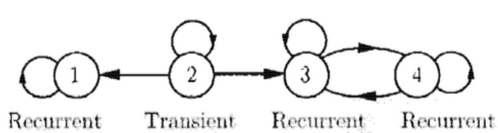

Figure 9.2: A Markov chain with recurrent and transient states (Figure 7.8 in the book).

State 1 is recurrent, because the only state that is accessible from 1 is itself, and obviously it is accessible from itself. This is actually a special recurrent state, sometimes called absorbing or terminal state — once we go to this state, we stay here forever.

State 3 is recurrent, because the states that are accessible from 3 are itself and 4, and we can go back to 3 from itself and 4. Similarly, state 4 is also recurrent.

State 2 is transient, because state 1 is accessible from 2, but 2 is *not* accessible from 1.