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ECE 342: Probability and Statistics **Spring 2025**

Lecture 8.2: Central Limit Theorem

Lecturer: Yuanzhang Xiao

Read BT Chapter 5.4.

Learning Objectives:

- Understand the Central Limit Theorem;
- Know how to use normal approximation to calculate probabilities.

8.1 The Central Limit Theorem

consider a sequence $X_1, X_2, \ldots$ of independent identically distributed (i.i.d.) random variables

- suppose that their mean is μ and variance is σ^2
- no other assumptions made (i.e., discrete or continuous, any CDF)

sum of first n of them:

$$S_n = X_1 + X_2 + \cdots + X_n$$

- $\mathbf{E}[S_n] = n\mu$ and $\mathrm{var}(S_n) = n\sigma^2$
- as n increases, the mean and variance go to $\infty \Rightarrow$ not a meaningful limit

sample mean of first n:

$$M_n = \frac{X_1 + \cdots + X_n}{n} = \frac{S_n}{n}$$

- $\mathbf{E}[M_n] = \mu$ and $\mathrm{var}(M_n) = \sigma^2/n$
- as n increases, the sample mean is almost a constant

is there anything in between the sum S_n and the sample mean M_n?

define

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

- $\mathbf{E}[Z_n] = 0$ and $\mathrm{var}(Z_n) = 1$

The Central Limit Theorem asserts that Z_n is "approximately" a standard normal random variable!

The Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common mean μ and variance σ^2, and define

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}.$$

Then the CDF of Z_n converges to the standard normal CDF

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx.$$

In other words, Z_n is approximately a standard normal random variable.

significance of the central limit theorem

- it holds for any i.i.d. random variables (i.e., discrete or continuous, arbitrary CDF)
- it provides one way to approximate a random variable using a standard normal random variable
  - only mean and variance (not CDF) of X_i are needed
- it implies that the sum S_n and the sample mean M_n can be approximated by normal random variables

Normal Approximation Based on The Central Limit Theorem

Let $S_n = X_1 + \cdots + X_n$, where X_i are independent identically distributed random variables with common mean μ and variance σ^2. If n is large, the probability $\mathbf{P}(S_n \le c)$ can be approximated by treating S_n as if it were normal, following the procedure below.

1. Calculate the mean $n\mu$ and the variance $n\sigma^2$ of S_n.
2. Calculate the normalized value $z = \frac{c - n\mu}{\sigma\sqrt{n}}$.
3. Use the approximation $\mathbf{P}(S_n \le c) \approx \Phi(z)$.

The same procedure can be generalized to other probabilities (e.g., $\mathbf{P}(S_n \ge c)$, $\mathbf{P}(a \le S_n \le b)$, etc.).

Exercises: Example 5.9, Example 5.10, Example 5.11 in Chapter 5 of BT.