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ECE 342: Probability and Statistics **Spring 2025**

Lecture 7.2: Cumulative Distribution Function

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Read BT Chapter 3.2.

Learning Objectives:

- Understand the concept of cumulative distribution functions (CDFs);
- Know the relationship between PDFs and CDFs;
- Know how to calculate the PDF from the CDF and calculate the CDF from the PDF.

7.1 Cumulative Distribution Functions

Cumulative Distribution Functions (CDF) of a random variable X

$$F_X(x) = \mathbf{P}(X \le x) = \begin{cases} \sum_{k \le x} p_X(k), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{x} f_X(t)dt, & \text{if } X \text{ is continuous.} \end{cases}$$

CDFs and PMFs are probabilities, while PMFs are *not* probabilities.

properties of a CDF:

- $F_X(x)$ is nondecreasing in x: if $x \le y$, then $F_X(x) \le F_X(y)$
- $F_X(x) \to 0$ when $x \to -\infty$, and $F_X(x) \to 1$ when $x \to \infty$
- if X is discrete, $F_X(x)$ is a piecewise constant function
- if X is continuous, $F_X(x)$ is a continuous function
- if X is discrete and takes integer values, we can get its PMF from its CDF by
$$p_X(k) = F_X(k) - F_X(k-1)$$
- if X is continuous, we can get its PDF from its CDF by
$$f_X(x) = \frac{dF_X}{dx}(x)$$

CDFs are often easier to calculate than PDFs.

- This is because the CDF represents probabilities and the PDF represents probability densities – it is easier to calculate probabilities than densities.
- We often calculate the CDF first and take its derivative to get the PDF.

Exercises: Examples 3.6, Problems 5, 6, 7 in Chapter 3 of BT.

Solution to Problem 5:
Denote the height of the triangle by h and the length of the base by b. We will see that the CDF and the PDF depend only on the height h and not on the shape of the triangle.
From the randomly chosen point, draw a line parallel to the base, and let A_x be the area of the triangle thus formed. The height of this triangle is $h-x$ and its base has length $b(h-x)/h$. Therefore, we have $A_x = b(h-x)^2/(2h)$. For $x \in [0,h]$, we have

$$F_X(x) = 1 - \mathbf{P}(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h-x)^2/(2h)}{bh/2} = 1 - \left(\frac{h-x}{h}\right)^2,$$

while $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > h$.
We can get the PDF by differentiating the CDF:

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \frac{2(h-x)}{h^2}, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 6:
Let X be the waiting time and Y be the number of customers found. Using the total probability theorem, we have

$$\begin{aligned} F_X(x) = \mathbf{P}(X \le x) &= \mathbf{P}(Y=0)\cdot\mathbf{P}(X \le x|Y=0) + \mathbf{P}(Y=1)\cdot\mathbf{P}(X \le x|Y=1) \\ &= \frac{1}{2}\cdot\mathbf{P}(X \le x|Y=0) + \frac{1}{2}\cdot\mathbf{P}(X \le x|Y=1) \\ &= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\left(1 - e^{-\lambda x}\right) \\ &= 1 - \frac{1}{2}e^{-\lambda x}, \quad \text{for } x \ge 0. \end{aligned}$$

Solution to Problem 7:
(a) The CDF of X is

$$F_X(x) = \mathbf{P}(X \le x) = \frac{\pi x^2}{\pi r^2} = \left(\frac{x}{r}\right)^2, \quad \text{for } x \in [0,r].$$

So the PDF is

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \frac{2x)}{r^2}, & \text{if } 0 \le x \le r, \\ 0, & \text{otherwise.} \end{cases}$$

The expectation is

$$\mathbf{E}[X] = \int_0^r x \cdot \frac{2x}{r^2}dx = \frac{2r}{3}.$$

The variance is

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \int_0^r x^2 \cdot \frac{2x}{r^2}dx - \left(\frac{2r}{3}\right)^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}.$$

(b) He gets a positive score in $[1/t, \infty)$ when $X \leq t$ and gets zero otherwise. Therefore, the CDF is $F_S(s) = 0$ for $s < 0$. For $s \in [0, 1/t]$, we have

$$F_S(s) = \mathbf{P}(\text{he hits outside the inner circle}) = 1 - \mathbf{P}(X \leq t) = 1 - \frac{t^2}{r^2}.$$

For $s \in (1/t, \infty)$, we use the total probability theorem to get

$$\begin{aligned} F_S(s) &= \mathbf{P}(S \leq s) = \mathbf{P}(X \leq t)\mathbf{P}(S \leq s|X \leq t) + \mathbf{P}(X > t)\mathbf{P}(S \leq s|X > t) \\ &= \frac{t^2}{r^2} \cdot \mathbf{P}(S \leq s|X \leq t) + \left(1 - \frac{t^2}{r^2}\right) \cdot \mathbf{P}(S \leq s|X > t) \end{aligned}$$

When the hit is outside the inner circle (i.e., $X > t$), the score is zero (i.e., $S = 0$). Therefore,

$$\mathbf{P}(S \leq s|X > t) = 1.$$

Moreover, we have

$$\mathbf{P}(S \leq s|X \leq t) = \mathbf{P}(1/X \leq s|X \leq t) = \frac{\mathbf{P}(1/s \leq X \leq t)}{\mathbf{P}(X \leq t)} = \frac{\frac{\pi t^2 - \pi(1/s)^2}{\pi r^2}}{\frac{\pi t^2}{\pi r^2}} = 1 - \frac{1}{s^2 t^2}.$$

Finally, we have

$$\mathbf{P}(S \leq s) = \frac{t^2}{r^2}\left(1 - \frac{1}{s^2 t^2}\right) + \left(1 - \frac{t^2}{r^2}\right) \cdot 1 = 1 - \frac{1}{s^2 r^2}.$$

Summarizing all of the above, the CDF of the score is is

$$F_S(s) = \begin{cases} 0, & \text{if } s < 0, \\ 1 - \frac{t^2}{r^2}, & \text{if } 0 \leq s \leq 1/t, \\ 1 - \frac{1}{s^2 r^2}, & \text{if } s > 1/t. \end{cases}$$