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ECE 342: Probability and Statistics **Spring 2025**

Lecture 7.1: Continuous Random Variables

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Read BT Chapter 3.1.

Learning Objectives:

- Understand the concept of PDFs of continuous random variables and its difference from PMFs of discrete random variables;
- Know how to calculate the expectation and the variance from the PDF.

7.1 Continuous Random Variables and PDFs

7.1.1 Probability Density Function

continuous random variable X – takes continuous values

the statistical properties of X are characterized by probability density function (PDF) $f_X(x)$

- PDF of X is a nonnegative function such that

$$\mathbf{P}(X \in B) = \int_B f_X(x)dx, \text{ for any subset } B \text{ of the real line}$$

- in particular, we have

$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

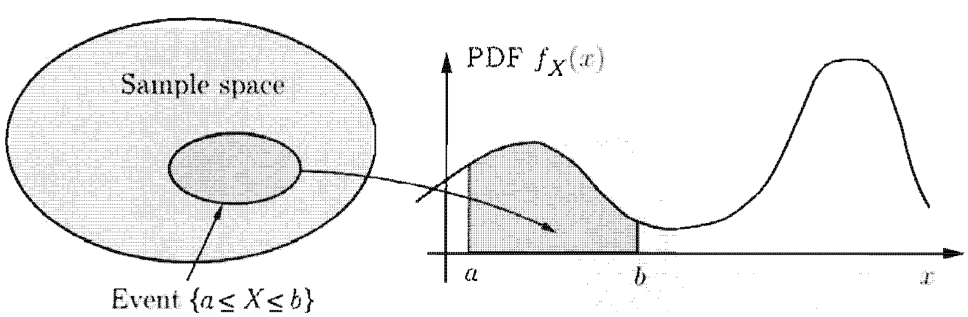

Figure 7.1: Illustration of a PDF (Figure 3.1 in the book).

a PDF should satisfy

- nonnegativity: $f_X(x) \geq 0$ for any x
- normalization: $\int_{-\infty}^{\infty} f_X(x)dx = 1$

note that PDF is NOT probability

- $\mathbf{P}(X = a) = 0 \Rightarrow$ it is not meaningful to talk about $\mathbf{P}(X = a)$ for continuous random variable X
- it is possible that $f_X(x) > 1$, but we need $\int_{-\infty}^{\infty} f_X(x)dx = 1$

then how to interpret PDF?

- PDF should be interpreted as density – "probability per unit length"

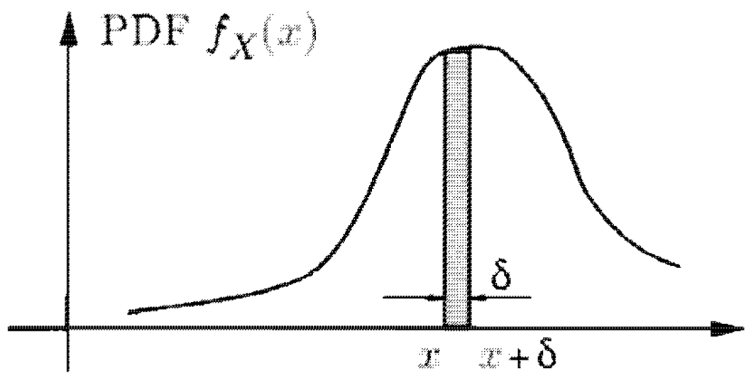

Figure 7.2: Interpretation of a PDF (Figure 3.2 in the book).

Exercises: Examples 3.1, 3.2, and 3.3 in Chapter 3 of BT.

7.1.2 Expectation

expectation / mean / expected value of a continuous random variable X

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx$$

- PMF $\rightarrow$ PDF, summation $\rightarrow$ integration

same definitions and properties as discrete random variables:

- expected value rule:
$$\mathbf{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$
- nth moment: $\mathbf{E}\left[X^n\right]$
- variance: $\mathrm{var}(X) \triangleq \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2$

- for linear function $Y = aX + b$, we have

$$\mathbf{E}[Y] = a\mathbf{E}[X] + b, \quad \mathrm{var}(Y) = a^2\mathrm{var}(X)$$

Exercises: Examples 3.4 and 3.5, Problems 1 and 2 in Chapter 3 of BT.

Solution to Problem 1:
The random variable Y is discrete and its PMF is

$$p_Y(1) = \mathbf{P}(X \le 1/3) = \int_0^{1/3} f_X(x)dx = \int_0^{1/3} 1dx = \frac{1}{3}$$
$$p_Y(2) = \mathbf{P}(X > 1/3) = \int_{1/3}^{1} f_X(x)dx = \int_{1/3}^{1} 1dx = \frac{2}{3}$$

Therefore, the expectation of Y, calculated from its PMF, is

$$E[Y] = 1 \times (1/3) + 2 \times (2/3) = \frac{5}{3}.$$

Alternatively, we can use the expected value rule as follows

$$E[Y] = \int_0^1 g(x)f_X(x)dx = \int_0^{1/3} 1 \cdot 1dx + \int_{1/3}^{1} 2 \cdot 1dx = \frac{5}{3}.$$

The two approaches give the same result.

Solution to Problem 2:
We can verify that

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{\lambda}{2}e^{-\lambda|x|}dx = 2 \cdot \frac{1}{2}\int_0^{\infty} \lambda e^{-\lambda x}dx = 2 \cdot \frac{1}{2} \cdot 1 = 1,$$

where we use the fact that $\int_0^{\infty} \lambda e^{-\lambda x}dx = 1$, namely the normalization property of the exponential PDF.
By symmetry, we have $E[X] = 0$.
For the variance, we first calculate

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^{\infty} x^2\lambda e^{-\lambda x}dx = \frac{2}{\lambda^2},$$

where we use the fact that the second moment of the exponential random variable is $2/\lambda^2$. Therefore, we have

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2}.$$