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**ECE 342: Probability and Statistics** **Spring 2025**

# Lecture 6.3: Independence of Random Variables

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Read BT Chapter 2.7.

## 6.1 Independence

we have learned about independence among events

similarly, we can define independence among random variables

### 6.1.1 Independence of a Random Variable from an Event

a random variable $X$ is independent of an event $A$ if

$$\mathbf{P}(X = x \text{ and } A) = \mathbf{P}(X = x) \cdot \mathbf{P}(A) = p_X(x) \cdot \mathbf{P}(A), \text{ for all } x,$$

or equivalently, when $\mathbf{P}(A) > 0$,

$$p_{X|A}(x) = p_X(x), \text{ for all } x.$$

**Exercises:** Examples 2.19 in Chapter 2 of BT.

### 6.1.2 Independence of Random Variables

two random variables $X$ and $Y$ are independent if

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y), \text{ for all } x, y,$$

or equivalently,

$$p_{X|Y}(x|y) = p_X(x), \text{ for all } x \text{ and for all } y \text{ such that } p_Y(y) > 0.$$

### 6.1.3 Conditional Independence of Random Variables

given an event $A$, two random variables $X$ and $Y$ are conditionally independent if

$$p_{X,Y|A}(x, y) = p_{X|A}(x) \cdot p_{Y|A}(y), \text{ for all } x, y,$$

or equivalently,

$$p_{X|Y,A}(x|y) = p_{X|A}(x), \text{ for all } x \text{ and for all } y \text{ such that } p_{Y|A}(y) > 0.$$

conditional independence and unconditional independence are not equivalent

### 6.1.4 Properties of Mean and Variance

if $X$ and $Y$ are independent, then

- $\mathbf{E}[XY] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$
- $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$
- $\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

the above properties do *not* hold true if $X$ and $Y$ are not independent

the above properties can be generalized to more than two random variables

**Exercises:** Examples 2.20 and 2.21, Problems 38–40 in Chapter 2 of BT.

**Solution to Problem 38:**
(a) Let $X$ be the number of red lights that Alice encouters. We know that $X$ is a binomial random variable with $n = 4$ and $p = 0.5$. Therefore, the mean is $E[X] = np = 2$ and the variance is $\mathrm{var}(X) = np(1-p) = 1$.
(b) The commuting time is the time under all green lights plus the total delay by red lights. The variance of the commuting time is then equal to the variance of the total delay. Since the total delay is $Y = 2X$, the variance is $\mathrm{var}(Y) = 4\mathrm{var}(X) = 4$.

**Solution to Problem 39:**
Let $X_1, X_2, \ldots, X_{10}$ be the numbers of eggs eaten in day 1, day 2, ..., day 10. Each $X_i$ is an uniform random variable between 1 and 6. Therefore, the mean is $E[X_i] = 3.5$ and the variance is

$$\mathrm{var}(X_i) = \frac{(b-a)(b-a+2)}{12} = \frac{(6-1)(6-1+2)}{12} = 2.92.$$

The total number of eggs is $X = X_1 + \cdots + X_{10}$.
Therefore, the mean is $E[X] = E[X_1] + \cdots + E[X_{10}] = 35$. Since $X_1, \ldots, X_{10}$ are independent, the variance is $\mathrm{var}(X) = \mathrm{var}(X_1) + \cdots + \mathrm{var}(X_{10}) = 29.2$.

**Solution to Problem 40:**
Define a success as a paper that receives a grade that has not been received before. Let $X_i$ be the number of papers between $i$th success and the $(i+1)$th success. Then the total number of papers is $X = 1+X_1+\cdots+X_5$. After receiving $i-1$ different grades so far ($i-1$ successes), each subsequent paper has probability $(6-i)/6$ of receiving a grade that has not been received before. Hence, each $X_i$ is a geometric random variable with $p = (6-i)/6$, and has a mean of $E[X_i] = 6/(6-i)$.
Therefore, we have

$$\begin{aligned} E[X] &= 1 + \sum_{i=1}^{5} E[X_i] \\ &= 1 + \sum_{i=1}^{5} \frac{6}{6-i} \\ &= 14.7 \end{aligned}$$