.2

ECE 342: Probability and Statistics **Spring 2025**

Lecture 6.2: Conditioning

Lecturer: Yuanzhang Xiao

Read BT Chapter 2.6.

TL;DR:

- We learned about conditional probability $\mathbf{P}(B|A)$ in Chapter 1.
- If $B = \{X = x\}$, we have $\mathbf{P}(\{X = x\}|A)$, denoted by $p_{X|A}(x)$ – conditional PMF given event A.
- If $A = \{Y = y\}$, we have $\mathbf{P}(\{X = x\}|\{Y = y\})$, denoted by $p_{X|Y}(x|y)$ – conditional PMF given $Y = y$.
- We can develop special versions of multiplication rule and total probability theorem.

We have learned about conditional probabilities: occurrence of an event affects probability of another event. Similarly, occurrence of an event can affect the statistics of random variables – conditional PMF.

6.1 Conditioning a Random Variable on an Event

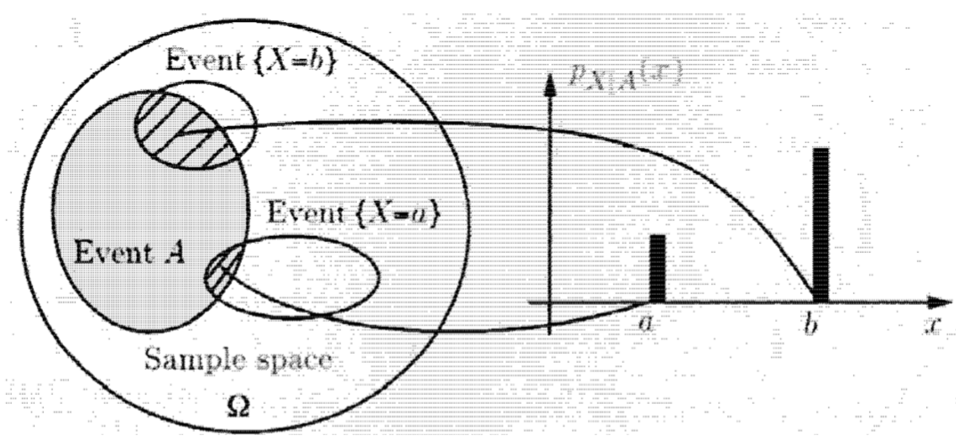

Figure 6.1: Illustration of conditional PMFs (Figure 2.12 in the book).

conditional PMF of X, conditioned on event A with $\mathbf{P}(A) > 0$, is

$$\boxed{p_{X|A}(x) = \mathbf{P}(X = x|A) = \frac{\mathbf{P}(\{X=x\}\cap A)}{\mathbf{P}(A)}}$$

we can verify that conditional PMF $p_{X|A}$ is a legitimate PMF

Exercises: Examples 2.12 and 2.13 in Chapter 2 of BT.

6.2 Conditioning a Random Variable on Another Random Variable

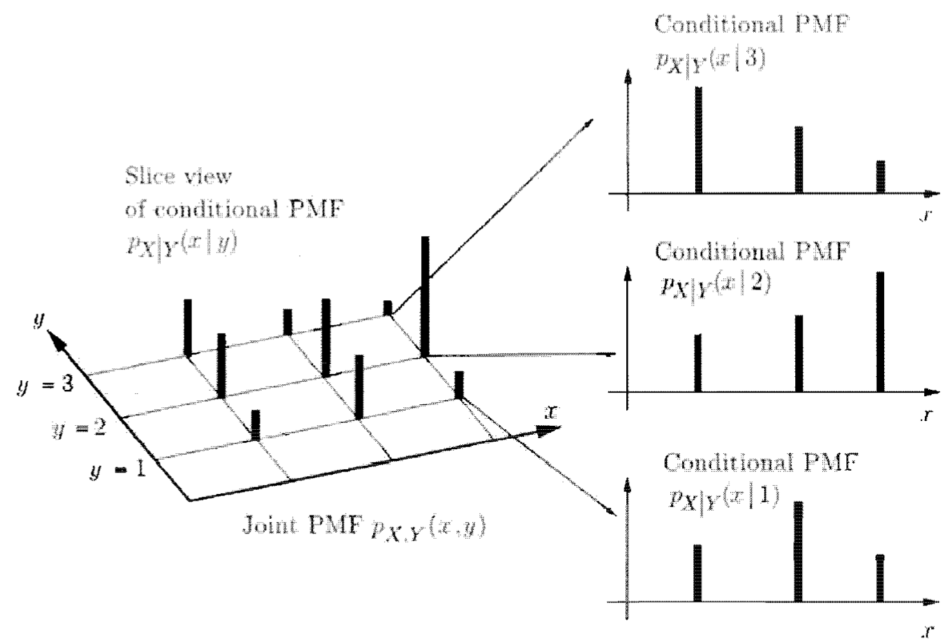

Figure 6.2: Illustration of conditional PMFs (Figure 2.13 in the book).

conditional PMF of X, conditioned on another random variable Y with $p(y) > 0$, is

$$p_{X|Y}(x|y) = \mathbf{P}(X = x|Y = y) = \frac{\mathbf{P}(X=x, Y=y)}{\mathbf{P}(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

we can verify that conditional PMF $p_{X|A}$ is a legitimate PMF

analogy to the multiplication rule:

$$p_{X,Y}(x,y) = p_Y(y) \cdot p_{X|Y}(x|y) = p_X(x) \cdot p_{Y|X}(y|x)$$

analogy to the total probability theorem:

$$p_X(x) = \sum_y p_{X,Y}(x,y) = \sum_y p_Y(y) \cdot p_{X|Y}(x|y)$$

Exercises: Examples 2.14 and 2.15, Problem 31 in Chapter 2 of BT.

Solution to Problem 31:
X is a binomial random variable with parameters $n = 4$ and $p = 1/6$. Given $X = x$, Y is a binomial random variable with $n = 4 - x$ and $p = 1/5$. Therefore, the joint PMF is

$$\begin{aligned} p_{X,y}(x,y) &= p_X(x) \cdot p_{Y|X}(y|x) \\ &= \left[\binom{4}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{4-x}\right] \cdot \left[\binom{4-x}{y}\left(\frac{1}{5}\right)^y\left(\frac{4}{5}\right)^{4-x-y}\right], \end{aligned}$$

for $0 \le x \le 4$ and $0 \le y \le 4 - x$.

6.3 Conditional Expectation

conditional expectation: expectation under the new PMF, namely conditional PMF, in the "new universe"

- conditional expectation of X given event A with $\mathbf{P}(A) > 0$:

$$\mathbf{E}[X|A] = \sum_x x \cdot p_{X|A}(x)$$

- conditional expectation of X given $Y = y$:

$$\mathbf{E}[X|Y = y] = \sum_x x \cdot p_{X|Y}(x|y)$$

expected value rule still holds:

$$\mathbf{E}[g(X)|A] = \sum_x g(X) \cdot p_{X|A}(x)$$

and

$$\mathbf{E}[g(X)|Y = y] = \sum_x g(X) \cdot p_{X|Y}(x|y)$$

analogy to the total probability theorem – the total expectation theorem:

- given a partition of the sample space $A_1, \ldots, A_n$:

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{P}(A_i) \cdot \mathbf{E}[X|A_i]$$

- given a random variable Y:

$$\mathbf{E}[X] = \sum_y p_Y(y) \cdot \mathbf{E}[X|Y = y]$$

Exercises: Examples 2.16 and 2.17, Problem 32 in Chapter 2 of BT.

Solution to Problem 32:
In this problem, we use the principle of divide and conquer (i.e., write S as a sum of multiple random variables) and the total expectation theorem.
For each couple i, we use binary random variables X_i and Y_i to denote the status of one partner and that of the other partner, respectively. Here, $X_i = 1$ or $Y_i = 1$ means that the person is alive, and $X_i = 0$ or $Y_i = 0$ means that the person is not alive.
Then we have

$$S = \sum_{i=1}^{m} X_i Y_i.$$

So the expectation is

$$\begin{aligned}
\mathbf{E}[S|A=a] &= \sum_{i=1}^{m} \mathbf{E}[X_i Y_i | A = a] \\
&= m\mathbf{E}[X_1 Y_1 | A = a] \\
&= m\left(\mathbf{E}[X_1 Y_1 | X_1 = 1, A = a]\mathbf{P}(X_1 = 1 | A = a) + \mathbf{E}[X_1 Y_1 | X_1 = 0, A = a]\mathbf{P}(X_1 = 1 | A = a)\right) \\
&= m\mathbf{E}[Y_1 | X_1 = 1, A = a]\mathbf{P}(X_1 = 1 | A = a) \\
&= m\left(1 \cdot \mathbf{P}[Y_1 = 1 | X_1 = 1, A = a] + 0 \cdot \mathbf{P}[Y_1 = 0 | X_1 = 1, A = a]\right)\mathbf{P}(X_1 = 1 | A = a) \\
&= m\mathbf{P}[Y_1 | X_1 = 1, A = a]\mathbf{P}(X_1 = 1 | A = a)
\end{aligned}$$

Since each person is equally likely to be alive, we have

$$\mathbf{P}(X_1 = 1 | A = a) = \frac{a}{2m}$$

and

$$\mathbf{P}[Y_1 | X_1 = 1, A = a] = \frac{a-1}{2m-1}.$$

Therefore, we have

$$\mathbf{E}[S|A=a] = m \cdot \frac{a}{2m} \cdot \frac{a-1}{2m-1} = \frac{a(a-1)}{2(2m-1)}.$$

Note that the expectation does not depend on p (the probability of being alive for each person)!