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ECE 342: Probability and Statistics **Spring 2025**

Lecture 5.1: Discrete Random Variables

Lecturer: Yuanzhang Xiao

Read BT Chapter 2.1, 2.2, and 2.3.

5.1 Discrete Random Variables

5.1.1 Basic Concepts

random variables – associate a **numerical value** with each outcome of the experiment

rigorous definition: a random variable is **a real-valued function of the experiment outcome**

$$X : \Omega \to \mathbb{R}$$

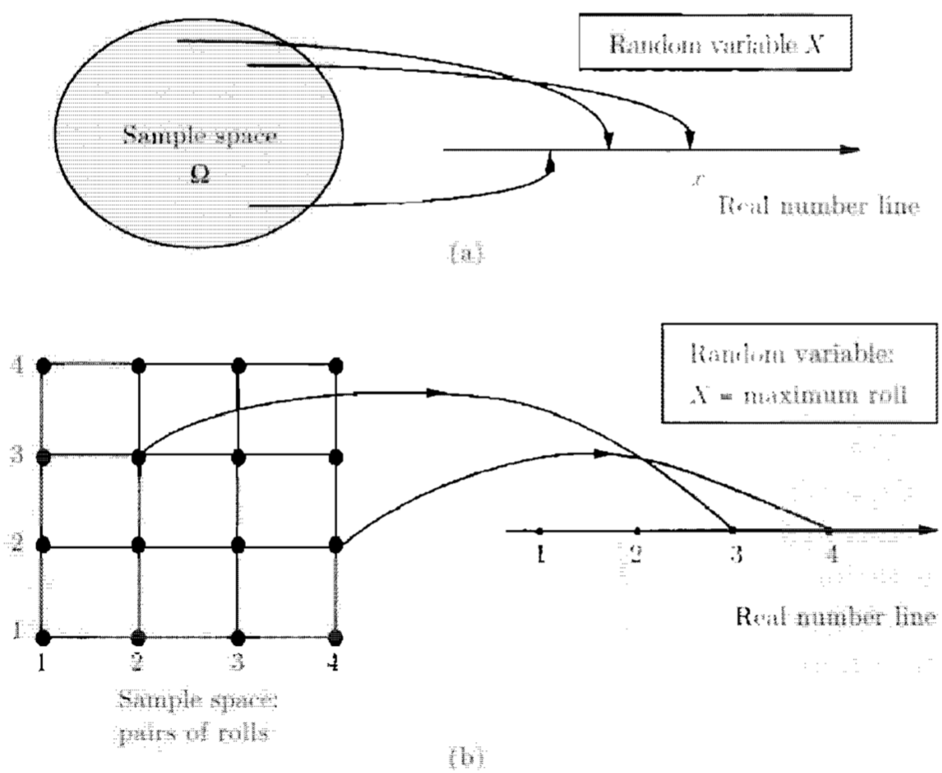

Figure 5.1: Illustration of random variables (Figure 2.1 in the book).

in this chapter, we focus on **discrete** random variables, whose values can be finite or countably infininte

check if the following objects are discrete random variables

- experiment: a sequence of 6 coin tosses
  - the number of heads in 6 tosses: discrete random variable
  - the results of 6 tosses: not a random variable
  - the number of heads minus the number of tails: discrete random variable
  - the number of heads divided by the number of tails: discrete random variable
- experiment: you wait for a bus, whose arrival time is random
  - your wait time: random variable, but not discrete
  - your wait time measured by your watch: discrete random variable
  - if you wait more than 20 minutes, you walk; whether you walk or not: not a random variable
  - the number of steps you would need to walk: discrete random variable

5.1.2 Probability Mass Function (PMF)

the statistical properties of a random variable are determined by its probability mass function (PMF)

$$p_X(x) = \mathbf{P}(\{X = x\}) = \mathbf{P}(X = x)$$

we use upper case letters to denote random variables (e.g., X), and lower case letters for its values (e.g., x)

properties of PMF:

- $\sum_x p_X(x) = 1$
- $\mathbf{P}(X \in S) = \sum_{x \in S} p_X(x)$

an example of two independent coin tosses:

- define the random variable X as the number of heads
- PMF of X is
$$p_X(x) = \begin{cases} 1/4, & \text{if } x = 0 \text{ or } x = 2 \\ 1/2, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$
- probability of at least one head is
$$\mathbf{P}(X > 0) = \sum_{x=1}^{2} p_X(x) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

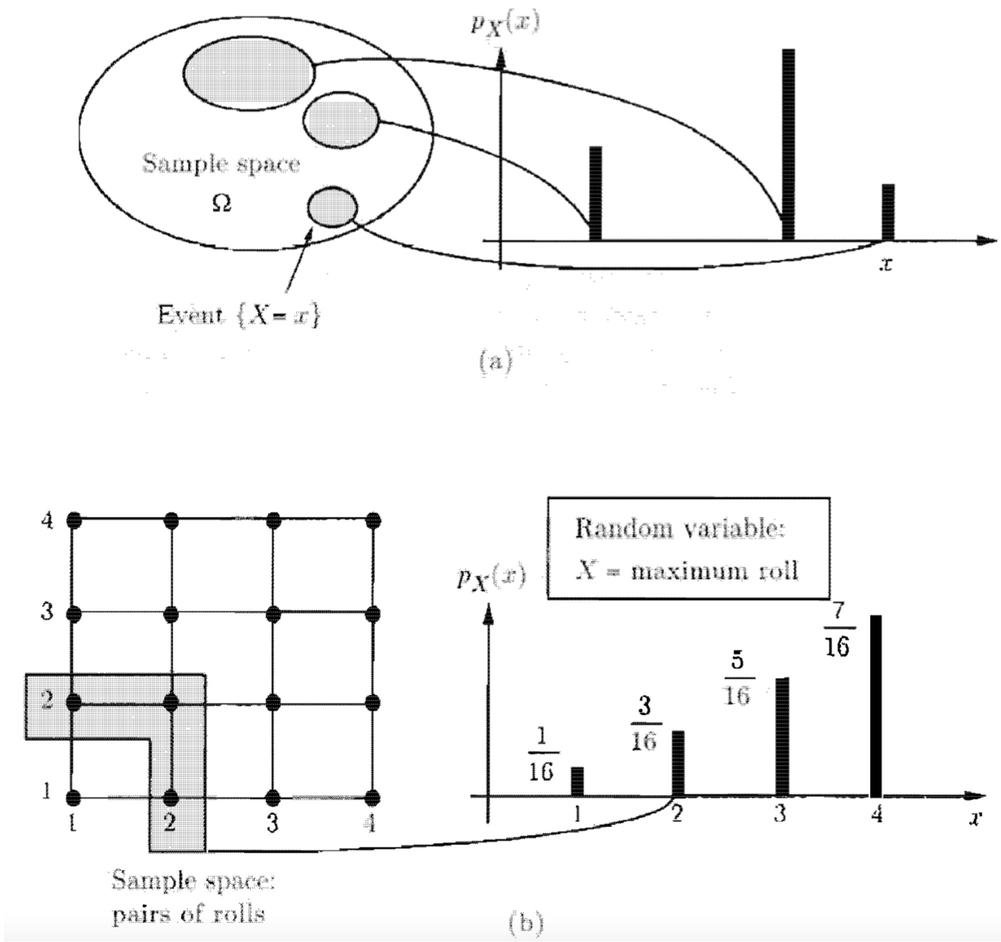

Figure 5.2: Illustration of the probability mass function and how to calculate it (Figure 2.2 in the book).

5.1.3 Some Special Random Variables

Bernoulli Random Variables

The Bernoulli random variable represents the binary outcomes of an experiment. For example,

- a coin toss: $X = 1$ if head, $X = 0$ if tail
- the test result: $X = 1$ if positive, $X = 0$ if negative

PMF of a Bernoulli random variable:

$$p_X(k) = \begin{cases} p, & \text{if } k = 1 \\ 1 - p, & \text{if } k = 0 \end{cases}$$

Binomial Random Variables

The binomial random variable is the number of certain outcomes in independent Bernoulli trials.

For example,

- the number of heads in n independent coin tosses
- the number of negative test results of n randomly and independently selected patients

PMF of a binomial random variable with parameters n and p:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \; k = 0, 1, \ldots, n,$$

where p is the probability of the outcome of interest in each Bernoulli trial

Geometric Random Variables

The geometric random variable is the "stopping time" of repeated independent Bernoulli trials.

For example,

- the number of coin tosses until a head comes up
- the number of tests until a positive test result is observed

PMF of a geometric random variable:

$$p_X(k) = (1-p)^{k-1} p, \; k = 1, 2, \ldots,$$

where p is the probability of "success" in each Bernoulli trial

Poisson Random Variables

The Poisson random variable represents the occurrence of rare events.

For example,

- the number of traffice accidents at an intersection during a week
- the number of patients with a rare disease in a population

PMF of a Poisson random variable:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k}, \; k = 0, 1, 2, \ldots,$$

where λ is the average number of occurrence of the event.

In some cases, Poisson random variables are approximations of binomial random variables

$$e^{-\lambda} \frac{\lambda^k}{k} \approx \binom{n}{k} p^k (1-p)^{n-k}$$

when $\lambda = np$, n is very large, and p is very small

Consider the number of car accidents in an area during a day. Suppose that the average number of accidents in a day is one. So we set $\lambda = 1$. Modeling the number of accidents as a Poisson random variable, the probability of 5 accidents is

$$\mathbf{P}(X_{\text{Poisson}} = 5) = e^{-1}\frac{1^5}{5!} = 0.00306.$$

Now we use binomial random variables (n, p) to model the number of accidents. We need to make sure that the average number of accidents is the same. In other words, we need to have $np = \lambda = 1$.

First, we divide the day into $n = 24$ hours. Then the probability of having an accident each hour is $p = 1/24$. The probability of 5 accidents is

$$\mathbf{P}(X_{\text{binomial}(24,1/24)} = 5) = \binom{24}{5} \cdot \left(\frac{1}{24}\right)^5 \left(\frac{23}{24}\right)^{19} = 0.00238.$$

Second, we divide the day into $n = 24 \times 60 = 1440$ minutes. Then the probability of having an accident each hour is $p = 1/1440$. The probability of 5 accidents is

$$\mathbf{P}(X_{\text{binomial}(1440,1/1440)} = 5) = \binom{1440}{5} \cdot \left(\frac{1}{1440}\right)^5 \left(\frac{1439}{1440}\right)^{1435} = 0.00305.$$

From this example, we can see that the binomial random variable and the Poisson random variable have similar PMFs when

- $\lambda = np$
- n is large and p is small

5.2 Functions of Random Variables

a function of a random variable, $Y = g(X)$, is also a random variables

PMF of Y is

$$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x)$$

Exercises: Example 2.1 in Chapter 2 of BT.

5.3 Exercises and Interview Problems

Exercises: Problems 1–7 in Chapter 2 of BT.

Below are some interview questions related to discrete random variables.

Google: Two teams play a series of games. The format is best of 7 – whoever wins 4 games first wins. Each team has a 50% chance of winning any game (no draws). What is the probability that the series goes to Game 7?

Goldman Sachs: Players A and B play a game where they take turns in flipping a biased coin, with p probability of landing on heads (and winning). Player A starts the game, and then the players pass the coin back and forth until one person flips heads and wins. What is the probability that A wins?